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Similarity of steady stratified flows

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Steady flows of an incompressible, inviscid, and non-diffusive fluid of variable density in a gravitational field are first considered. By a transformation it is shown conclusively that there are infinitely many flows with the same flow pattern, provided the density gradients of these flows at any section (e.g. far upstream) differ only by a multiplicative constant. These flows have identical local internal Froude numbers at all corresponding points of the flows and, hence, identical local Richardson numbers. They are therefore dynamically similar. Every time a solution for one stratification is obtained, one has in fact obtained the solutions for infinitely many stratifications.

The creation of vorticity in steady stratified flows is then examined, and it is shown that this creation can be divided into two parts, one part being entirely due to the inertial effect and the other originating from the gravity effect of density variation.

Finally, compressibility is considered and the results on similarity of stratified flows and on vorticity and circulation are extended to apply to steady flows of gases stratified in entropy.

1. Similarity of steady stratified flows of an incompressible fluid

For an incompressible and non-diffusive fluid stratified in density, the equation of incompressibility is

$$D\rho/Dt = 0, \tag{1}$$

where, since only steady flows are considered,

$$D/Dt = u_\alpha \partial/\partial x_\alpha. \tag{2}$$

In (1) and (2), ρ is the density, x_1, x_2, and x_3 are Cartesian co-ordinates, and u_1, u_2, and u_3 are the corresponding velocity components. The summation convention is used in (2). The equation of continuity is, by virtue of (1),

$$\partial u_i/\partial x_i = 0. \tag{3}$$

If the fluid is also assumed inviscid, the equations of motion are

$$\rho u_\alpha \, \partial u_i/\partial x_\alpha = -\partial p/\partial x_i - g\rho\delta_{i3} \quad (i = 1, 2, 3), \tag{4}$$

where p is the pressure, g is the gravitational acceleration acting in the direction of decreasing x_3, and δ_{i3} is the Kronecker delta.

Let the density be put in the form

$$\rho = \rho_0 + \rho_1(x_1, x_2, x_3), \tag{5}$$

where ρ_0 is a constant, and consider another stratified fluid with the density distribution

$$\hat{\rho} = \hat{\rho}_0 + \hat{\rho}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3), \tag{6}$$

where $\hat{\rho}_0$ is another constant and the circumflexes on the x's denote the co-ordinates for the flow with density $\hat{\rho}$. Since two flows can be similar only if the geometry of the boundaries are similar, we denote the length scales of the two flows by L and $\hat{L}$, and write

$$m = \hat{L}/L. \tag{7}$$

A point in the flow with length scale L is said to be corresponding to a point in the flow with length scale $\hat{L}$ if the dimensionless co-ordinates (measured in units of L or $\hat{L}$) of the two points are identical. For dynamical similarity to exist, we must have

$$\hat{\rho}_1/\rho_1 = r \tag{8}$$

at corresponding points of the two flows, r being a positive constant.

The question is then posed: Can a flow have the density distribution ρ and be similar to a given flow with the density distribution $\hat{\rho}$? We shall show that the answer is in the affirmative.

If the solution with distribution $\hat{\rho}$ has velocity components $\hat{u}_i$, we have

$$\hat{\rho}\hat{u}_\alpha\,\partial\hat{u}_i/\partial\hat{y}_\alpha = -\partial\hat{\pi}/\partial\hat{y}_i - \hat{L}g\hat{\rho}_1\delta_{i3}, \tag{9}$$

where

$$\hat{\pi} = \hat{p} + \hat{\rho}_0 g x_3, \quad \hat{y}_i = x_i/\hat{L}. \tag{10}$$

We also have

$$\hat{u}_\alpha\,\partial\hat{\rho}/\partial\hat{y}_\alpha = 0 \tag{11}$$

and

$$\partial\hat{u}_i/\partial\hat{y}_i = 0. \tag{12}$$

Now let (actually an arbitrary constant can be added to π or $\hat{\pi}$)

$$u_i = (\hat{\rho}/rm\rho)^{\frac{1}{2}}\hat{u}_i, \quad \pi = \hat{\pi}/rm, \quad y_i = x_i/L, \tag{13}$$

where π is defined by

$$\pi = p + \rho_0 g x_3. \tag{14}$$

Then, the first equation in (13), and (5), (6), and (8) guarantee that

$$u\,\partial\rho/\partial y_\alpha = 0, \tag{15}$$

provided (11) is satisfied. Furthermore, obviously

$$u_\alpha\,\partial\hat{\rho}/\partial y_\alpha = 0. \tag{16}$$

Equations (7), (8) and (13) allow us to write†

$$\rho u_\alpha\,\partial u_i/\partial y_\alpha = (\hat{\rho}/rm)\,\hat{u}_\alpha\,\partial\hat{u}_i/\partial\hat{y}_\alpha. \tag{17}$$

Thus (9) can be written as

$$rm\rho u_\alpha\,\partial u_i/\partial y_\alpha = -rm\,\partial\pi/\partial y_i - r\hat{L}g\rho_1\delta_{i3}, \tag{18}$$

† Remember that $y_i = \hat{y}_i$ at corresponding points, so that $\partial/\partial y_\alpha = \partial/\partial\hat{y}_\alpha$. This would be even clearer if we had used $\hat{x}_i = mx_i$.

which, after division by rm, is

$$\rho u_\alpha \partial u_i/\partial y_\alpha = -\partial\pi/\partial y_i - Lg\rho_1\delta_{i3}. \quad (19)$$

This is exactly (4) with the Cartesian co-ordinates in dimensionless form. Thus, if (9) is satisfied by $\hat{u}_i$ and $\hat{\pi}$, u_i and π given by (13) satisfy (19), or (4).

Also, because of (15) and (16), (3) is satisfied if (12) is. Hence we have proved what we set out to prove. The boundary conditions, if they are kinematical, are identical since the boundary geometries are identical, and, if they are dynamical (such as at density discontinuities), are also identical since dynamic boundary conditions are natural boundary conditions derivable from the differential equations. Hence the boundary conditions are satisfied by the flow (u_1, ρ, p) if they are satisfied by the flow $(\hat{u}_i, \hat{\rho}, \hat{p})$.

If we define the local internal Froude number F at any point of the flow by

$$F^2 = \rho u_\alpha u_\alpha / g|\nabla\rho| L^2 \quad (20)$$

(and similarly for the flow with density $\hat{\rho}$), and the local Richardson number Ri at any point of the flow by

$$Ri = g|\nabla\rho|/\rho|\nabla q|^2, \quad q^2 = u_\alpha u_\alpha, \quad (21)$$

(and similarly for the flow with density $\hat{\rho}$), then we can say that the two flows have identical local internal Froude numbers at corresponding points (i.e. for $y_i = \hat{y}_i$), and consequently the same local Richardson numbers at these points. The flow patterns are also similar, by virtue of the first equation in (13). The two flows are indeed similar geometrically, kinematically, and dynamically.

But since ρ_0 and r are arbitrary, we are not merely treating one flow similar to the flow $\hat{\rho}$; we are treating a doubly infinite family of flows, all of which are similar to the flow for $\hat{\rho}$, and hence to one another. This result is new, and I think it is very useful for laboratory simulation of natural phenomena. Note that density discontinuities are not ruled out. But, for similarity to exist between any two flows, they must occur at corresponding places, and, wherever they occur, their ratio must be the same constant (denoted by r in this paper). We note that when the density variation is small as compared with the mean density, the factor $(\hat{\rho}/\rho)^{\frac{1}{2}}$ in (13) can be replaced by $(\hat{\rho}_0/\rho_0)^{\frac{1}{2}}$ (and by 1 if water is used in the laboratory to model lakes or oceans), and the effect of density variation, important at low Froude numbers, is embodied in the factor r and is entirely associated with gravity.

In conclusion, to ensure similarity, the requirements expressed by (8) and the first equation in (13) must be satisfied at corresponding sections somewhere, say far upstream. Note that ρ does not have to be proportional to $\hat{\rho}$ at corresponding points.

2. Creation of vorticity in steady stratified flows of an incompressible fluid

Let ρ_0 now denote some constant reference density. We shall use the transformation (Yih 1958)

$$u_i' = (\rho/\rho_0)^{\frac{1}{2}} = u_i, \quad (22)$$

and study the creation of vorticity. With the vorticity vector ξ_i defined by

$$\boldsymbol{\xi} = \operatorname{curl}\mathbf{u}, \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3), \quad \mathbf{u} = (u_1, u_2, u_3), \quad (23)$$

we define the vorticity of the associated velocity $\mathbf{u}' \equiv (u_1', u_2', u_3')$, and

$$\boldsymbol{\xi}' = \operatorname{curl} \mathbf{u}'. \tag{24}$$

The Euler equations can be written as

$$\rho_0 u_\alpha' \,\partial u_i'/\partial x_\alpha = -\partial p/\partial x_i - \rho\, \partial \Omega/\partial x_i, \tag{25}$$

where Ω is the body-force potential, and the equation of continuity can be written as

$$\partial u_i'/\partial x_i = 0. \tag{26}$$

From (25) and (26), by cross-differentiation, we obtain

$$\rho_0 (\mathbf{u}'.\nabla)\boldsymbol{\xi}' = \rho_0 (\boldsymbol{\xi}'.\nabla)\mathbf{u}' - \nabla\rho \times \nabla\Omega. \tag{27}$$

The first term in (27) is the substantial derivative of the vorticity vector, the second term gives the effects of stretching and turning on the three components of $\boldsymbol{\xi}'$, so that the last term represents the rate of creation of the vorticity $\boldsymbol{\xi}'$ (or the components of it) as a result of density variation in the presence of a gravitational field. It is always a horizontal vector. The true vorticity is then created in the following two ways:

(i) by the inertial effect of the density variation, through the transformation (22); for instance, if gravity were absent, a steady flow originating from a large reservoir, where the velocity is zero, would have zero $\boldsymbol{\xi}'$, which means $\boldsymbol{\xi}$ would be created – by inertia alone if ρ is not uniform.

(ii) by the creation of $\boldsymbol{\xi}'$ by density variation in the presence of a gravitational field. In general (i.e. except for the unlikely case $\boldsymbol{\xi}' \neq 0$ but $\boldsymbol{\xi} = 0$) vorticity will be created. This creation is attributed to the action of gravity on a stratified fluid.

Note that although the pseudo-vorticity created is always horizontal, it does not necessarily remain so because of the turning of the pseudo-vorticity lines.

Since the pseudo-vorticity $\boldsymbol{\xi}'$ is associated with the pseudo-circulation Γ' defined by

$$\Gamma' = \oint u_i' \, dx_i, \tag{28}$$

we can profitably consider the rate of change of Γ' around a closed material circuit. Let D/Dt stand for the substantial derivative. Then, using (25) and

$$(D/Dt)\, dx_i = du_i, \tag{29}$$

we have

$$\frac{D}{Dt}\Gamma' = \oint \left\{ \left(-\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\rho}{\rho_0}\frac{\partial \Omega}{\partial x_i} \right) dx_i + d(\tfrac{1}{2}u_\alpha u_\alpha) \right\} = \oint \frac{\Omega}{\rho_0} d\rho, \tag{30}$$

showing clearly that Γ' would be constant if gravity were not present, and that gravity can change the pseudo-circulation Γ' in a stratified fluid and through it change the true circulation Γ defined by

$$\Gamma = \oint u_i \, dx_i.$$

In the absence of gravity, Γ' would remain constant, but Γ would in general change with time – as a result of the inertial effect of density variation.

The creation of vorticity and circulation in a nonhomogeneous fluid is the subject of Bjerknes' theorems, of course. But since these theorems apply to unsteady flows as well as steady flows, their content is necessarily less far-reaching: they say that the creation of vorticity or circulation arises from a term containing the factor

$$\rho^{-2}\nabla p \times \nabla \rho,$$

but can say no more about the constitution of p. By restricting attention to steady flows, I have been able, in effect, to separate p into a dynamical part which accounts for the inertial effect of density variation, and a static part which accounts for the gravity effect of the constant part of the density. As a corollary of (30), we see that, if the material circuit is taken on a constant density surface,

$$D\Gamma'/Dt = 0 \quad \text{and} \quad D\Gamma/Dt = 0,$$

which implies that any vorticity created lies in the constant density surface (by the use of Stokes' theorem).

3. Extension to entropy stratification in gases

The results obtained above will now be extended to steady flows of compressible fluids stratified in entropy. We shall consider ideal gases and denote the ratio of the specific heat at constant pressure (c_p) to that at constant volume by the usual symbol γ, and we shall write, with the subscript c denoting some constant reference quantity,

$$\lambda = (\rho/\rho_c)/(p_c/p)^{\gamma}, \tag{31}$$

which is equal to a constant times $\exp(-S/c_p)$, S being the entropy. Since the flow is steady and heat conduction and viscous dissipation are neglected, the entropy does not change along a streamline, and we have

$$u_\alpha \,\partial\lambda/\partial x_\alpha = 0. \tag{32}$$

Again, we consider two λ-distributions:

$$\hat{\lambda} = \hat{\lambda}_0 + \hat{\lambda}_1, \tag{33}$$

$$\lambda = \lambda_0 + \lambda_1, \tag{34}$$

where the λ_0 and $\hat{\lambda}_0$ are assumed constant and

$$r = \hat{\lambda}_1/\lambda_1 \tag{35}$$

at corresponding points of the two flows, r being a positive constant. And, again, the question is posed: Can a flow have the entropy distribution given by λ and be similar to a given flow with the entropy distribution given by $\hat{\lambda}$? The answer is again in the affirmative but, instead of (13), the transformation demonstrating this is (with L, $\hat{L}$, y_i, y, and $\hat{y}_i$ retaining their meanings as before and with m given by (7))

$$u_i = (\hat{\lambda}/rm\lambda)^{\frac{1}{2}}\hat{u}_i, \quad \pi = \hat{\pi}/rm, \tag{36}$$

where π and $\hat{\pi}$ are defined by

$$\pi = \int dp/\rho' + \lambda_0 g x_3, \quad \hat{\pi} = \int d\hat{p}/\hat{\rho}' + \hat{\lambda}_0 g x_3, \tag{37}$$

and

$$\rho' = \rho/\lambda, \quad \hat{\rho}' = \hat{\rho}/\hat{\lambda}. \tag{38}$$

The demonstration that the flow (u_i, π, λ) satisfies the equations of motion, the equation of continuity, and (32) is quite similar to the corresponding demonstration given above for incompressible fluids. It is sufficient to recall the results of Yih (1960 or 1980, p. 6). In particular, from (31),

$$\rho'/p^\gamma = \rho_c'/p_c^\gamma = \text{constant}, \tag{39}$$

so that the integrals in (37) exist.

Again, given geometrical similarity, dynamical similarity is assured if at some section far upstream (33), (34), and (35) hold, and the velocity distributions for the two flows are related according to the first equation in (36).

As to the creation of circulation and vorticity, we note that (Yih 1960), with

$$u_i' = \lambda^{\frac{1}{2}} u_i, \quad \rho' = \rho/\lambda, \quad p = p', \tag{40}$$

we have

$$\partial(\rho' u_\alpha')/\partial x_\alpha = 0, \tag{41}$$

and

$$\rho' u_\alpha' \partial u_i'/\partial x_\alpha = -\partial p'/\partial x_i - g\rho\delta_{i,3}. \tag{42}$$

Since (39) is satisfied, by cross differentiation of (42) after division by ρ' we have

$$(\mathbf{u}'.\nabla)(\boldsymbol{\xi}'/\rho') = (\boldsymbol{\xi}'/\rho').\nabla\mathbf{u}' - \nabla\lambda \times \nabla\Omega, \tag{43}$$

where Ω is the body-force potential gx_3. Again, the last term is a horizontal vector, and represents the creation of pseudo-vorticity. The pseudo-vorticity, however, is not necessarily horizontal, because of the turning of the pseudo-vorticity lines.

If Γ' is again defined by (28), with u_i' given by (40), we have, instead of (30),

$$D\Gamma'/Dt = \oint \Omega\, d\lambda, \tag{44}$$

showing that $D\Gamma'/Dt$ is zero if gravity does not exist or if the material circuit along which Γ' is taken lies on a surface of constant entropy. Equation (44) shows the effect of the last term in (43), as (30) shows the effect of the last term in (27).

These are again two means of creation of true vorticity, as in the case of the incompressible fluid, and we shall not repeat the nearly identical statements.

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